October Math Gems

PROBLEM OF THE WEEK 7

§1 Problems

Problem 1.1. The real root of the equation $8x^3 - 3x^2 - 3x - 1 = 0$ can be written in the form $\frac{\sqrt[3]{a}+\sqrt[3]{b}+1}{c}$, where a, b, and c are positive integers. Find $a + b + c$.

Problem 1.2. Let m be the largest real solution to the equation

$$\frac{3}{x-3} + \frac{5}{x-5} + \frac{17}{x-17} + \frac{19}{x-19} = x^2 - 11x - 4.$$

There are positive integers a, b, c such that $m = a + \sqrt{b + \sqrt{c}}$. Find $a + b + c$.

Problem 1.3. In a Martian civilization, all logarithms whose bases are not specified are assumed to be base b, for some fixed $b \geq 2$. A Martian student writes down

$$3\log(\sqrt{x}\log x) = 56$$

$$\log_{\log(x)}(x) = 54$$

and finds that this system of equations has a single real number solution $x > 1$. Find b.

Problem 1.4. Let r, s, and t be the three roots of the equation

$$8x^3 + 1001x + 2008 = 0.$$

Find $(r+s)^3 + (s+t)^3 + (t+r)^3$.

Problem 1.5. Real numbers x and y satisfy the equation $x^2 + y^2 = 10x - 6y - 34$. What is $x + y$?

Problem 1.6. What is the value of

$$\frac{2^{2014} + 2^{2012}}{2^{2014} - 2^{2012}}?$$

Problem 1.7. For certain real numbers a, b, and c, the polynomial

$$g(x) = x^3 + ax^2 + x + 10$$

has three distinct roots, and each root of $g(x)$ is also a root of the polynomial

$$f(x) = x^4 + x^3 + bx^2 + 100x + c.$$

What is $f(1)$?

Problem 1.8. Positive integers a and b satisfy the condition

$$\log_2(\log_{2^a}(\log_{2^b}(2^{1000}))) = 0.$$

Find the sum of all possible values of $a + b$.

Problem 1.9. A hexagon that is inscribed in a circle has side lengths 22, 22, 20, 22, 22, and 20 in that order. The radius of the circle can be written as $p + \sqrt{q}$, where p and q are positive integers. Find $p + q$.

Problem 1.10. What is the sum of all possible values of k for which the polynomials $x^2 - 3x + 2$ and $x^2 - 5x + k$ have a root in common?

Problem 1.11. If $y + 4 = (x - 2)^2, x + 4 = (y - 2)^2$, and $x \neq y$, what is the value of $x^2 + y^2$?

Problem 1.12. Points $(\sqrt{\pi}, a)$ and $(\sqrt{\pi}, b)$ are distinct points on the graph of $y^2 + x^4 = 2x^2y + 1$. What is $|a - b|$?

Problem 1.13. Suppose that a, b, and c are positive real numbers such that $a^{\log_3 7} = 27$, $b^{\log_7 11} = 49$, and $c^{\log_{11} 25} = \sqrt{11}$. Find

$$a^{(\log_3 7)^2} + b^{(\log_7 11)^2} + c^{(\log_{11} 25)^2}.$$

Problem 1.14. How many positive integers n satisfy

$$\frac{n + 1000}{70} = \lfloor \sqrt{n} \rfloor?$$

(Recall that $\lfloor x \rfloor$ is the greatest integer not exceeding x.)

Problem 1.15. In rectangle $ABCD$, $AB = 6$, $AD = 30$, and G is the midpoint of $\overline{AD}$. Segment AB is extended 2 units beyond B to point E, and F is the intersection of $\overline{ED}$ and $\overline{BC}$. What is the area of $BFDG$?

Problem 1.16. Find $ax^5 + by^5$ if the real numbers a, b, x, and y satisfy the equations

$$\begin{aligned} ax + by &= 3, \\ ax^2 + by^2 &= 7, \\ ax^3 + by^3 &= 16, \\ ax^4 + by^4 &= 42. \end{aligned}$$

Problem 1.17. Given that $\sin a + \sin b = \frac{1}{10}$ and $\cos a + \cos b = \frac{1}{9}$, find $\lfloor \tan^2(a + b) \rfloor$

Problem 1.18. In a triangle ABC, D is midpoint of BC . If $\angle ADB = 45°$ and $\angle ACD = 30°$, determine $\angle BAD$.

Problem 1.19. Find $a + 2b + 3c$ If

$$a + \frac{3}{b} = 3$$

$$b + \frac{2}{c} = 2$$

$$c + \frac{1}{a} = 1$$

Problem 1.20. Find the minimum value of

$$f(x) = \frac{x^2 + x + 1}{x^2 + 2x + 1}$$

for all x in the domain of $f(x)$
